REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

1. AGENCY USE ONLY *(Leave blank)*

2. REPORT DATE
96/3/18

3. REPORT TYPE AND DATES COVERED
Final Technical - 12/92 to 11/95

4. TITLE AND SUBTITLE
Solving Large and Dense Eigenvalue Problems that Arise in Physics

5. FUNDING NUMBERS
N00014-93-1-0268

6. AUTHOR(S)
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7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
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8. PERFORMING ORGANIZATION REPORT NUMBER
A-10415

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)
ONR Code 251 Attn JWK, Ballston Tower One
800 North Quincy Street
Arlington, VA 22217-5660

10. SPONSORING/MONITORING AGENCY REPORT NUMBER

11. SUPPLEMENTARY NOTES

12a. DISTRIBUTION/AVAILABILITY STATEMENT
Approved for public release; distribution is unlimited

12b. DISTRIBUTION CODE

13. ABSTRACT *(Maximum 200 words)*
We worked on a fundamental problem of decomposing a signal into a small set of decaying complex exponentials. This problem arises in a wide range of disciplines, including nuclear magnetic resonance, speech processing and system identification. We developed a new class of numerical algorithms, and gave a simple, purely linear algebraic proof on why our new approach works. Our class contains two arbitrary matrices F and G. Specific choices of these two matrices result in Prony's and Kung's methods. So all our theoretical results cover the two procedures. This advance is important, for Kung's proof can be difficult to digest. Other choices of F and G give rise to new methods with other desirable characteristics; e.g., our new Hankel QRD method is about ten times faster than Kung's scheme, also known as the Hankel SVD method. Another attraction of the QRD approach is that it is easily updatable to accommodate new data, which is not so for an SVD technique.

19960717 020

14. SUBJECT TERMS
signal enhancement, harmonic retrieval, matrix pencil, denoising

15. NUMBER OF PAGES
9 (total)

16. PRICE CODE

17. SECURITY CLASSIFICATION OF REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
UNCLASSIFIED	UNCLASSIFIED	UNCLASSIFIED	UL

NSN 7540-01-280-5500

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Rensselaer

Department of Computer Science

March 18, 1996

Scientific Officer Code: 1111MA
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Office of Naval Research
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In accordance with Attachment Number 2 under Grant No. N00014-93-1-0268, enclosed please find my Final Technical Report. Please contact me if you have any questions.

Thank you very much for supporting my work. I really appreciate it.

Sincerely,

Frank

Franklin T. Luk
Principal Investigator

Enclosures (3)

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1. Problem of Interest

We worked on a fundamental problem:

- Decompose a signal into a small set of decaying complex exponentials.

Usually, a limited sequence $\{s_k\}$ of noisy samples is available and the physics of the problem implies that the noiseless exponentials are damped sinusoids. The formal signal model is

$$s_k = \sum_{r=1}^{\mathrm{R}} a_r z_r^{k-1}, \tag{1}$$

where the coefficients $\{a_r\}$ and knots $\{z_r\}$ are $2\mathrm{R}$ unknowns to be determined. This problem arises in a wide range of disciplines, including nuclear magnetic resonance [1], speech processing [3], and system identification [4]. Two effective solution procedures were given by Prony in 1795 [5] and S.Y. Kung *et al.* in 1983 [4]. Unfortunately, Prony's method is relatively unknown and Kung's method and associated proof are tightly bound to the application in system identification.

1.1. Our Contributions

We developed a new class of numerical algorithms for solving this exponential approximation problem. Our class includes both methods of Prony and Kung as special cases. We gave a simple, purely linear algebraic proof on why our new approach works.

2. Novel Matrix Pencil Approach

The samples $\{s_k\}$ are laid out as entries in a Hankel matrix:

$$H^{\mathrm{K}\times\mathrm{L}} \equiv \begin{pmatrix} s_1 & s_2 & s_3 & \cdots & s_{\mathrm{L}} \\ s_2 & s_3 & s_4 & \cdots & s_{\mathrm{L}+1} \\ s_3 & s_4 & s_5 & \cdots & s_{\mathrm{L}+2} \\ \vdots & \vdots & \vdots & & \vdots \\ s_{\mathrm{K}} & s_{\mathrm{K}+1} & s_{\mathrm{K}+2} & \cdots & s_{\mathrm{K}+\mathrm{L}-1} \end{pmatrix},$$

where $\mathrm{K} > \mathrm{R}$. MatLab-like notations are used for matrix indexing, e.g., the colon ':' indicates a range (hence $A_{3,:}$ refers to the third row of A). We devised a new matrix pencil procedure for computing $\{z_r\}$, after which we would solve a special (called Yule-Walker) set of linear equations to find $\{a_r\}$. For full generality, we introduce two arbitrary but nonsingular transformations F and G, where F is $(\mathrm{K}-1) \times (\mathrm{K}-1)$ and G is $\mathrm{L} \times \mathrm{L}$. Define two new $(\mathrm{K}-1) \times \mathrm{L}$ matrices $C^{[1]}$ and $C^{[2]}$ by

$$C^{[1]} \equiv FH_{1:\mathrm{K}-1,:}G \qquad \text{and} \qquad C^{[2]} \equiv FH_{2:\mathrm{K},:}G.$$

Consider the matrix pencil problem:

$$C^{[2]}y = \zeta C^{[1]}y.$$

We proved that solutions exist such that Y is a $\mathrm{L} \times \mathrm{R}$ matrix of linearly independent eigenvectors:

$$C^{[2]}Y = C^{[1]}YD_\zeta^{\mathrm{R}}, \tag{2}$$

where D_ζ^{R} denotes a $\mathrm{R} \times \mathrm{R}$ diagonal matrix of eigenvalues $\{\zeta_l\}$. In addition, we also proved that $\{\zeta_l\} = \{z_r\}$.

Matrix Pencil Method.

Step 1. Choose matrices F and G.

Step 2. Solve the matrix pencil: $C^{[2]}Y = C^{[1]}YD_\zeta^{\mathrm{R}}$. Set $\{\zeta_l\} = \{z_r\}$.

2.1. Our Contributions

The important result is that $\{\zeta_l\} = \{z_r\}$, so that eigenvalues of the matrix pencil provide the required solution. A huge benefit is the freedom to choose F and G. Specific choices result in Prony's and Kung's methods. Other choices of F and G could give us new methods with other desirable characteristics.

3. Full Column Rank Case

Both Prony's and Kung's methods assume full column rank, i.e., L = R. Hence the matrix Y is $\mathrm{R} \times \mathrm{R}$ and nonsingular and (2) becomes $C^{[2]} = C^{[1]}(YD_\zeta^{\mathrm{R}}Y^{-1})$. Define $X \equiv YD_\zeta^{\mathrm{R}}Y^{-1}$. We see that X satisfies

$$C^{[1]}X = C^{[2]}. \tag{3}$$

The matrix equation (3) is usually overdetermined; in the presence of noise, X may be solved by least-squares or total least-squares methods. The matrix pencil method simplifies to an eigenvalue procedure.

Eigenvalue Method.

Step 1. Choose matrices F and G.

Step 2. Find X by solving the linear equations: $C^{[1]}X = C^{[2]}$.

Step 3. Find an eigenvalue decomposition of X: $X = YD_\zeta^{\mathrm{R}}Y^{-1}$. Set $\{\zeta_l\} = \{z_r\}$.

3.1. Prony's Procedure.

Prony further assumed that $\mathrm{K} = \mathrm{R} + 1$. Hence the matrices $C^{[1]}$ and $C^{[2]}$ are $\mathrm{R} \times \mathrm{R}$. Choose $F = I^{\mathrm{R}\times\mathrm{R}}$ and $G = H^{-1}_{1:\mathrm{R},1:\mathrm{R}}$. So $C^{[1]} = I^{\mathrm{R}\times\mathrm{R}}$ and and (3) simplifies to $X = C^{[2]}$. Note that

$$C^{[2]} = H_{2:\mathrm{R}+1,1:\mathrm{R}}\,H^{-1}_{1:\mathrm{R},1:\mathrm{R}} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -\gamma_0 & -\gamma_1 & -\gamma_2 & \cdots & -\gamma_{\mathrm{R}-2} & -\gamma_{\mathrm{R}-1} \end{pmatrix},$$

where the $\{\gamma_i\}$ satisfy the Yule-Walker equation. Hence $C^{[2]}$ is a companion matrix to the polynomial $p_{\mathrm{R}}(z)$, defined by $p_{\mathrm{R}}(z) = \gamma_0 + \gamma_1 z + \cdots + \gamma_{\mathrm{R}-1}z^{\mathrm{R}-1} + z^{\mathrm{R}}$.

Prony's Method.

Step 1. Choose $F = I^{\mathrm{R}\times\mathrm{R}}$ and $G = H^{-1}_{1:\mathrm{R},1:\mathrm{R}}$.

Step 2. Solve the Yule-Walker equation.

Step 3. Find the roots $\{z_r\}$ of the Prony polynomial $p_{\mathrm{R}}(z)$. Set $\{\zeta_l\} = \{z_r\}$.

3.2. Kung's Procedure

Start by computing a singular value decomposition (SVD) of a $\mathrm{K} \times \mathrm{R}$ Hankel matrix H: $H = U\Sigma V^H$. Choose $F = I^{(\mathrm{K}-1)\times(\mathrm{K}-1)}$ and $G = V\Sigma^{-1}$. Then $C^{[1]} = U_{1:\mathrm{K}-1,:}$ and $C^{[2]} = U_{2:\mathrm{K},:}$. So (3) reduces to $U_{1:\mathrm{K}-1,:}X = U_{2:\mathrm{K},:}$. An important advantage of Kung's method over Prony's procedure is that the polynomial $p_{\mathrm{R}}(z)$ is not explicitly computed.

Kung's Method.

Step 1. Choose $F = I^{(\mathrm{K}-1)\times(\mathrm{K}-1)}$ and $G = V\Sigma^{-1}$. Compute an SVD of H: $H = U\Sigma V^H$.

Step 2. Solve the matrix equation: $U_{1:\mathrm{K}-1,:}X = U_{2:\mathrm{K},:}$.

Step 3. Find an eigenvalue decomposition of X: $X = YD_\zeta^{\mathrm{R}}Y^{-1}$. Set $\{\zeta_l\} = \{z_r\}$.

3.3. Our Contributions

We showed that our matrix pencil scheme includes both Prony's and Kung's methods as special cases. So all our theoretical results, including the purely linear algebraic proof on why our new method works, cover these two procedures too. This is an important advance, for Kung's proof [4] can be difficult to digest.

4. New Methods

We may replace the expensive SVD by other techniques. The only restriction is that we need to operate on H from the left side. An example of a much cheaper, and yet almost as accurate, technique is the QR decomposition (QRD). Compute a QR decomposition of a $\mathrm{K} \times \mathrm{R}$ Hankel matrix H: $H = QR$. Choose $F = I^{(\mathrm{K}-1)\times(\mathrm{K}-1)}$ and $G = R^{-1}$. We find that $C^{[1]} = Q_{1:\mathrm{K}-1,:}$ and $C^{[2]} = Q_{2:\mathrm{K},:}$. So (3) reduces to

$$Q_{1:\mathrm{K}-1,:}X = Q_{2:\mathrm{K},:} \ . \tag{4}$$

Hankel QRD Method.

Step 1. Choose $F = I^{(\mathrm{K}-1)\times(\mathrm{K}-1)}$ and $G = R^{-1}$.

Step 2. Solve the linear equations: $Q_{1:\mathrm{K}-1,:}X = Q_{2:\mathrm{K},:}$.

Step 3. Find an eigenvalue decomposition of X: $X = YD_{\zeta}^{\mathrm{R}}Y^{-1}$. Set $\{\zeta_l\} = \{z_r\}$.

4.1. Our Contributions

Our new Hankel QRD method is about ten times faster than Kung's scheme (also known as the Hankel SVD method). An attractive property of the QRD approach is that it is easily updatable to accommodate new data, which is not so for an SVD technique.

5. Denoising

Our problem is particularly sensitive to errors in the samples $\{s_k\}$, and a preliminary noise removal procedure is all-important. We assume an additive error model:

$$H = H^{[s]} + H^{[n]},$$

where $H^{[s]}$ and $H^{[n]}$ denote the signal and noise components of H, respectively. The SVD of the signal matrix H provides a convenient and numerically stable way to determine the noise subspace. Indeed, it often suffices to find a "gap" in the entries on the diagonal of Σ. Unfortunately, the Hankel structure of H is almost certainly lost in $H^{[s]}$. Cadzow [2] proposed to iterate signal extraction with the SVD followed by a restoration of the Hankel structure via averaging along the antidiagonals. He showed that under relatively mild assumptions this procedure will converge and reported very good noise removal performance.

5.1. Our Contributions

First, we constructed a counter-example for Cadzow's method, and showed that it will converge to a Hankel matrix of the wrong rank.

Second, we proposed a new approach based on the following decomposition:

$$H = V^{\mathrm{K}\times\mathrm{M}}(z)D_a^{\mathrm{M}}V^{\mathrm{L}\times\mathrm{M}}(z)^T, \tag{5}$$

where $V^{\mathrm{K}\times\mathrm{M}}(z)$ and $V^{\mathrm{L}\times\mathrm{M}}(z)$ are respectively $\mathrm{K}\times\mathrm{M}$ and $\mathrm{L}\times\mathrm{M}$ Vandemonde matrices, and the middle matrix D_a^{M} is diagonal. If $\mathrm{K} \geq \mathrm{M}$, $\mathrm{L} \geq \mathrm{M}$ and all the $\{z_r\}$ values are distinct, then the Hankel matrix H has exact rank M. We examined algorithms to determine the factorization (5) for a given Hankel matrix. It seems reasonable to attempt to synthesize another Hankel matrix $\hat{H}$ of rank R, where $\mathrm{R} < \mathrm{M}$, from zeroing out $(\mathrm{M}-\mathrm{R})$ of the $\{a_r\}$ values. However, selecting which a_r to set to zero is a nontrivial task. Unlike singular vectors, the columns of a Vandemonde matrix are not orthonormal. Hence, it is not always appropriate to select the R largest (in magnitude) a_r's to synthesize a good approximation to H. Our work is still in progress.

6. Combined Technique

In our overall computation, we chose to grossly "overestimate" the rank R of the available samples. Just as in Prony's method, we would pick R so that the matrices were square. Figure 1 shows a cloud of zeros from the resulting polynomial for an example where the numerical rank of the signal was five, but a 65×65 noisy system had been solved. This example is taken from Van Huffel [6]; the signal-to-noise ratio is approximately 2.4 (not dB).

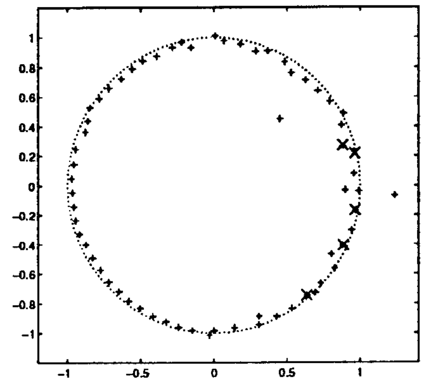

Figure 1. Prony's method with 65 degrees of freedom on data with numerical rank 5. The '+' are the computed z values while the exact values are depicted with '×'.

With the common assumption that the noise space is approximately orthogonal to the signal subspace, we would expect that a few components will still fit the embedded signal while most other zeros serve as outlets to fit the noise. Our numerical experiments seem to confirm this conjecture. Given our stability criterion, it is logical to discard those zeros (of the polynomial) that are greater than one in magnitude and hence cannot correspond to a stable component of the signal. Figure 2 shows the result.

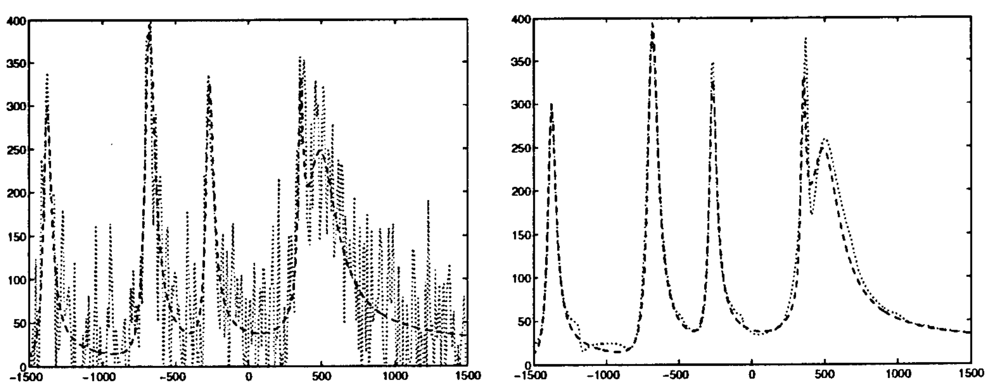

Figure 2. **left:** The discrete Fourier spectrum of the given noisy signal. **right:** The spectrum after removing the unstable z values and computing the new samples s. The dashed curve shows the noiseless spectrum in both plots.

6.1. Our Contributions

There is a significant difference between our approach and Prony's method: we fix N, the number of samples, and choose $\text{R} = \lfloor \text{N}/2 \rfloor$, whereas Prony fixed R, the rank of the matrix, and chose $\text{N} = 2\text{R}$. With noise, the Hankel matrix we get is always nonsingular. Our scheme has the important advantage that we do not need to decide on R at the very beginning. Another benefit is that the resultant Yule-Walker is square and can be solved using any of the well known techniques.

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